JOURNAL OF APPROXIMATION THEORY 34, 384–396 (1982)

Approximation of Continuous Functions by Generalized Favard Operators

W. GAWRONSKI AND U. STADTMÜLLER

Abteilung für Mathematik, Universität Ulm, Oberer Eselsberg, D-7900 Ulm, West Germany

Communicated by P. L. Butzer

Received March 27, 1981

0. INTRODUCTION

It is well known (e.g., [8; 15]) that classical Bernstein polynomials, generalized Bernstein polynomials, Baskakov operators, and other similar operators approximate a function f with order $O(1/n)$ provided the derivative f' belongs to the class Lip 1 and satisfies a certain growth condition in case of an unbounded interval. These operators are discrete, linear, and positive. More precisely, they are of the form

$$F_n(f; x) = \sum_j f\left(\frac{j}{n}\right) p_{jn}(x), \qquad f \in C(I), \tag{0.1}$$

where $I \subseteq \mathbb{R}$ denotes throughout an interval, and the functions p_{jn} satisfy $p_{jn}(x) \geqslant 0$, $x \in I$, $j \in \mathbb{Z}$, $n \in \mathbb{N}$. More generally, the above order of approximation holds if $\{p_{jn}(x)\}_{j=-\infty}^{\infty}$ is the n-fold convolution of a probability lattice distribution with expectation x (see also [1; 4; 11, Chap. 7; 17]).

To increase the rate of convergence (provided f is sufficiently smooth) we mention two methods. By forming suitable linear combinations of operators F_n (e.g., [14]) or by taking iterates of Fejér–Korovkin type for F_n [10; 12] the approximation order becomes $O(n^{-r})$ when $f \in C_{2r}(I)$, $r \in \mathbb{N}$. But for $r \geqslant 2$ the resulting operators are no longer positive.

Recently Butzer and Wehrens [6], and Swetits and Wood [16], among others, have derived discrete, positive polynomial operators of Bernstein type for approximating continuous functions on $[-1, 1]$; these operators are of the form

$$L_n(f; x) = \sum_{j=0}^{2n} f(x_{j,2n})\, \phi_{j,2n}(x), \qquad f \in C[-1, 1], \tag{0.2}$$

$\phi_{j,2n}$ being positive polynomials on $[-1, 1]$. They showed that the

0021-9045/82/040384–13$02.00/0

corresponding saturation order is $1/n^2$. Despite the advantage that L_n is a polynomial operator it should be pointed out that (0.2) involves the zeros $x_{j,2n} \in [-1, 1]$ of the $2n$th Legendre polynomial and Cotes numbers, quantities which can be computed only approximately. This is due to discretization of an integral operator by means of the Gauss–Christoffel quadrature formula.

In this paper we construct a *discrete* and *positive* operator (on $C(\mathbb{R})$) of type (0, 1), using *equidistant nodes* j/n, $j \in \mathbb{Z}$, and for p_{jn} the "discrete Gaussian kernel"

$$g_{jn}(x) := \frac{1}{\sqrt{2\pi}\, n\sigma_n} \exp\left(-\frac{(j-nx)^2}{2n^2\sigma_n^2}\right), \tag{0.3}$$

$j \in \mathbb{Z}$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, $\sigma_n > 0$. This choice is suggested by Favard's approach [9] of discretizing the singular integral of Gauss–Weierstrass

$$W(f; x; \sigma) := \frac{1}{\sqrt{2\pi}\, \sigma} \int_{-\infty}^{\infty} f(t)\, e^{-(t-x)^2/2\sigma^2}\, dt, \qquad \sigma > 0 \tag{0.4}$$

by means of

$$F_n(f; x) = \sum_j f\left(\frac{j}{n}\right) g_{jn}(x) \tag{0.5}$$

with $\sigma_n^2 = \lambda/2n$ (λ being a positive constant) which is also its approximation order [2; 3]. Therefore we consider the *generalized Favard operator* (0.5), allowing σ_n^2 to be a more general null sequence. Using a functional equation for a certain theta series (e.g., [7, pp. 63, 64; 5, p. 126; 3]) we obtain the central result of this paper giving the asymptotic relation

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} (F_n(f; x) - f(x)) = \frac{f''(x)}{2}, \qquad x \in \mathbb{R} \tag{0.6}$$

for $f \in C_2(\mathbb{R})$ with a suitable growth condition. It turns out that except for a constant factor the choice $\sigma_n^2 = \log n/n^2$ is best possible in (0.6). Moreover, (0.6) holds for the truncated version of $F_n(f; x)$, namely,

$$F_n^*(f; x) := \frac{1}{\sqrt{2\pi}\, n\sigma_n} \sum_{|j-nx| \leqslant c_n} f\left(\frac{j}{n}\right) \exp\left(-\frac{(j-nx)^2}{2n^2\sigma_n^2}\right), \tag{0.7}$$

where $n\sigma_n \sqrt{\log 1/\sigma_n} = o(c_n)$, thereby showing that $O(\sigma_n^2)$ is the saturation order for F_n and F_n^* (Section 2). In the case $\sigma_n^2 = \log n/n^2$ a comparison with the polynomial operator (0.2) shows that the computation of the *finite exponential operator* F_n^* requires "$(\log n)^{1+\varepsilon}$ terms" asymptotically ($\varepsilon > 0$) and the approximation order differs by a factor $\log n$ only. On the other hand, it is not difficult to see that the order of approximation by means of general positive operators of type (0.1) cannot be made better than $O(1/n^2)$ in a certain sense (Section 1).

In addition we prove a saturation theorem for the operators in question (Section 2).

1. Preliminary Results

In this section we collect some auxiliary results. First consider the general positive operator (0.1) and suppose that F_n maps $C(I)$ into itself. Then it is known from the general theory of positive linear operators that the approximation of continuous functions is governed by the "central moments," defined by

$$\begin{aligned} r_{0,n}(x) &:= F_n(1;x) - 1 = \sum_j p_{jn}(x) - 1, \\ r_{\nu,n}(x) &:= F_n((t-x)^\nu; x) = \sum_j \left(\frac{j}{n} - x\right)^\nu p_{jn}(x) \qquad x \in I, \quad \nu \in \mathbb{N} \end{aligned} \tag{1.1}$$

(e.g., [8, Chap. 2]). Assuming the existence of the sums in (1.1) at least for $\nu = 0, 1, 2$, we prove ($[x]$ denotes, as customary the largest integer not exceeding x).

LEMMA 1. (i) *If* $\sup_{x\in I} r_{\nu,n}(x)$ *is finite for* $\nu = 0, 2$, *then*

$$\sup_{x\in I} r_{2,n}(x) \geqslant \frac{1}{4n^2}\left(1 + \sup_{\substack{j\in\mathbb{Z} \\ (1/n)(j+1/2)\in I}} r_{0,n}\left(\frac{1}{n}\left(j+\frac{1}{2}\right)\right)\right).$$

(ii) *The particular choice* ($I = \mathbb{R}$)

$$p_{jn}(x) = \bar{p}_{jn}(x) := \begin{cases} nx - [nx], & j = [nx] + 1 \\ 1 - nx + [nx], & j = [nx] \\ 0, & \text{otherwise} \end{cases}$$

gives $r_{0,n}(x) = r_{1,n}(x) \equiv 0$ *and*

$$\sup_{x\in\mathbb{R}} r_{2,n}(x) = \frac{1}{4n^2}.$$

Proof. (i) For every $n \in \mathbb{N}$ there exists a real number $x_n \in I$ such that

$$\left| \frac{j}{n} - x_n \right| \geqslant \frac{1}{2n}, \qquad \text{for all} \quad j \in \mathbb{Z}; \tag{1.2}$$

e.g., $x_n := (m + 1/2)/n \in I$ satisfies (1.2) for suitable $m \in \mathbb{Z}$. Hence we have, since $p_{jn} \geqslant 0$,

$$\begin{aligned}\sup_{x \in I} r_{2,n}(x) &\geqslant \frac{1}{4n^2} \sup_{x_n \in I} \sum_j p_{jn}(x_n) \\ &= \frac{1}{4n^2} \left(1 + \sup_{\substack{m \in \mathbb{Z} \\ (1/n)(m+1/2) \in I}} r_{0,n}\left(\frac{1}{n}\left(m + \frac{1}{2}\right)\right)\right).\end{aligned}$$

(ii) The relations $r_{0,n}(x) = r_{1,n}(x) \equiv 0$ are trivial and

$$\begin{aligned}\sup_{x \in \mathbb{R}} r_{2,n}(x) &= \sup_{x \in \mathbb{R}} \sum_j \left(\frac{j}{n} - x\right)^2 \bar{p}_{jn}(x) \\ &= \frac{1}{n^2} \max_{x \in \mathbb{R}} (nx - [nx] - (nx - [nx])^2) = \frac{1}{4n^2}.\end{aligned}$$

It is known from the general theory of approximation by positive linear operators that the quality of approximation of continuous functions is determined by the largest of the quantities $r_{\nu,n}(x)$, $\nu = 0, 1, 2$ (e.g., [8, Chaps. 2, 5]). Hence part (i) of Lemma 1 indicates that the approximation order of the operators (0.1) in general cannot be better than $O(1/n^2)$ provided $r_{\nu,n}(x) \to 0$ as $n \to \infty$, $\nu = 0, 1, 2$, and the function f is not linear (compare also Theorem 2). In many cases we have $r_{\nu,n} = o(r_{2,n})$ as $n \to \infty$, $\nu = 0, 1$ and therefore the order of approximation is given by $r_{2,n}$. In the classical examples mentioned at the beginning of the Introduction $r_{0,n}(x) = r_{1,n}(x)$ are even $\equiv 0$.) The particular choice $p_{jn} = \bar{p}_{jn}$ in Lemma 1(ii) leads to an operator of type (0.1) having the saturation order

$$\frac{1}{n^2} (nx - [nx])(1 - nx + [nx])$$

with maximum $1/4n^2$. Thus the sequence $\bar{p}_{jn}(x)$ leads to an optimal operator (0.1) in the sense just indicated, but for obvious reasons the functions p_{jn} are desired to be "smooth" and "elementary." To this end in the sequel we extend Favard's approach of discretizing the singular integral of Gauss–Weierstrass (0.4) and consider the case $p_{jn} = g_{jn}$ defined in (0.3) which leads to the operator F_n in (0.5).

Using a well-known functional equation and the Fourier series expansion of a certain theta series we start with the identity (e.g., [5, p. 126; 7, pp. 63, 64; 3])

$$\sum_j g_{jn}(x) = \frac{1}{\sqrt{2\pi}\, n\sigma_n} \sum_{j=-\infty}^{\infty} \exp\left(-\frac{(j-nx)^2}{2n^2\sigma_n^2}\right)$$
$$= \sum_{\nu=-\infty}^{\infty} e^{-2\pi^2\nu^2 n^2\sigma_n^2} e^{2\pi i\nu xn}$$
$$= 1 + 2\sum_{\nu=1}^{\infty} e^{-2\pi^2\nu^2 n^2\sigma_n^2} \cos(2\pi\nu xn). \tag{1.3}$$

Further we put

$$b_n := \exp(-2\pi^2 n^2 \sigma_n^2). \tag{1.4}$$

Differentiating (1.3) with respect to x the following estimates for the moments $r_{\nu,n}$ defined in (1.1) are readily verified.

LEMMA 2. *If $\sigma_n > 0$ for $n \geqslant n_0$, then*

$$|r_{0,n}(x)| \leqslant \frac{2b_n}{1-b_n}, \qquad |r_{1,n}(x)| \leqslant \frac{4\pi n\sigma_n^2 b_n}{(1-b_n)^2},$$

$$|r_{2,n}(x) - \sigma_n^2| \leqslant \sigma_n^2 |r_{0,n}(x)| + 8\pi^2 n^2 \sigma_n^4 \frac{b_n(1+b_n)}{(1-b_n)^3}$$

for all $x \in \mathbb{R}$ and $n \geqslant n_0$. Furthermore we have

$$r_{4,n}(x) = O(\sigma_n^8 + n^4\sigma_n^8 b_n), \qquad n \to \infty,$$

uniformly in $x \in \mathbb{R}$ provided $\sigma_n \to 0$, but $n\sigma_n \to \infty$.

Actually a refined inspection of (1.3) and its derivatives shows that except for constants the estimates in Lemma 2 are asymptotically sharp. But the results in its present form are sufficient for our purpose.

Looking at the Bohman–Korovkin theorems [e.g., [8. Chap. 2]) we demand $r_{\nu,n}(x) \to 0$ as $n \to \infty$, $\nu = 0, 1, 2$, in order to have uniform approximation on compact intervals. By the first part of Lemma 2 this forces $\sigma_n \to 0$ and $n\sigma_n \to \infty$ as $n \to \infty$. To obtain a good rate of convergence $r_{0,n}$, $r_{1,n}$, and $r_{2,n}$ nearly have to be of the same order of magnitude. Under this postulation we choose σ_n^2 as

$$\bar{\sigma}_n^2 := \frac{\log n}{n^2} \tag{1.5}$$

leading below to a rate of convergence being optimal in the sense just indicated. (Taking into account multiplicative constants the choice $\sigma_n^2 = \log n/\pi^2 n^2$ would be optimal.) We exhibit this case by specializing Lemma 2.

LEMMA 2′. *If $\sigma_n^2 = \bar{\sigma}_n^2 = \log n/n^2$, then*

$$|r_{0,n}(x)| \leqslant \frac{3}{n^{2\pi^2}}, \qquad |r_{1,n}(x)| \leqslant \frac{13 \log n}{n^{2\pi^2+1}},$$

$$\left| r_{2,n}(x) - \frac{\log n}{n^2} \right| \leqslant \frac{81 \log^2 n}{n^{2\pi^2+2}}$$

for all $x \in \mathbb{R}$ and $n \geqslant 2$; moreover

$$r_{4,n}(x) = O\left(\frac{\log^4 n}{n^8}\right)$$

uniformly in $x \in \mathbb{R}$ as $n \to \infty$.

Next, we estimate the discrepancy between F_n and F_n^* (see (0.3), (0.5), and (0.7)). In the sequel suppose that $\{c_n\}$ is a sequence of positive numbers satisfying

$$\frac{c_n}{n\sigma_n} =: a_n \uparrow \infty, \qquad n\sigma_n \uparrow \infty. \tag{1.6}$$

Further we introduce the truncated moments by ($x \in \mathbb{R}$)

$$r_{0,n}^*(x) := F_n^*(1; x) - 1 = \sum_{|j-nx| \leqslant c_n} g_{jn}(x) - 1 \tag{1.7}$$

and

$$r_{\nu,n}^*(x) := F_n^*((t-x)^\nu; x) = \sum_{|j-nx| \leqslant c_n} \left(\frac{j}{n} - x\right)^\nu g_{jn}(x), \qquad \nu \in \mathbb{N}, \tag{1.8}$$

and prove

LEMMA 3. (i) *Suppose that $\sigma_n > 0$ and $(c_n - 1)/n\sigma_n \geqslant \sqrt{2}$ for $n \geqslant n_0$. Then we have for $x \in \mathbb{R}$ and $n \geqslant n_0$*

$$|r_{0,n}^*(x)| \leqslant \frac{2b_n}{1-b_n} + \sqrt{\frac{2}{\pi}} \frac{n\sigma_n}{c_n - 1} \exp\left(-\frac{1}{2}\left(\frac{c_n - 1}{n\sigma_n}\right)^2\right) =: \alpha_n,$$

$$|r_{1,n}^*(x)| \leqslant \frac{4\pi n\sigma_n^2 b_n}{(1-b_n)^2} + \sqrt{\frac{2}{\pi}} \sigma_n \exp\left(-\frac{1}{2}\left(\frac{c_n - 1}{n\sigma_n}\right)^2\right) =: \beta_n,$$

$$|r^*_{2,n}(x)-\sigma_n^2| \leqslant \frac{2\sigma_n^2 b_n}{1-b_n} + 8\pi^2 n^2 \sigma_n^4 \frac{b_n(1+b_n)}{(1-b_n)^3}$$
$$+\sqrt{\frac{2}{\pi}}\sigma_n^2\left(\frac{c_n-1}{n\sigma_n}+\frac{n\sigma_n}{c_n-1}\right)\exp\left(-\frac{1}{2}\left(\frac{c_n-1}{n\sigma_n}\right)^2\right) =: \gamma_n,$$

and

$$r^*_{4,n}(x) = O(\sigma_n^4) \qquad \textit{holds uniformly in } x\in\mathbb{R},$$
$$\textit{as } n\to\infty, \textit{ provided } \sigma_n\to 0.$$

(ii) *Assume that* $f:\mathbb{R}\to\mathbb{R}$ *satisfies the growth condition*

$$|f(x)| \leqslant M e^{K|x|}, \qquad x\in\mathbb{R}, \tag{1.9}$$

for some constants $M, K\geqslant 0$. *Then for all* n, *with* $\sigma_n>0$ *and* $(c_n-1)/n\sigma_n > K\sigma_n$

$$|F_n(f;x)-F_n^*(f;x)|$$
$$\leqslant \sqrt{\frac{2}{\pi}}\frac{M\exp(K|x|+K^2\sigma_n^2/2)}{(c_n-1)/n\sigma_n - K\sigma_n}\exp\left(-\frac{1}{2}\left(\frac{c_n-1}{n\sigma_n}-K\sigma_n\right)^2\right)$$

holds for all $x\in\mathbb{R}$.

Proof. (i) For $\nu=0,1,2$ we have

$$r^*_{\nu,n}(x) = r_{\nu,n}(x) - \sum_{|j-nx|>c_n}\left(\frac{j}{n}-x\right)^{\nu} g_{jn}(x)$$
$$=: r_{\nu,n}(x) - R_{\nu,n}(x).$$

Thus we obtain with $\xi_{j,n} := (j-nx)/n\sigma_n$

$$|R_{\nu,n}(x)| \leqslant \sqrt{\frac{2}{\pi}}\sigma_n^{\nu}\sum_{\xi_{j,n}>a_n} e^{-\xi_{j,n}^2/2}\xi_{j,n}^{\nu}(\xi_{j+1,n}-\xi_{j,n}).$$

If $\xi\geqslant\sqrt{2}$, then $e^{-\xi^2/2}\xi^{\nu}$ ($\nu=0,1,2$) is decreasing and thus we can estimate the sum by an integral.

By (1.6), we obtain for $n\geqslant n_0$

$$|R_{\nu,n}(x)|$$
$$\leqslant \sqrt{\frac{2}{\pi}}\sigma_n^{\nu}\sum_{\mu=-1}^{\infty}\int_{(c_n+\mu)/n\sigma_n}^{(c_n+\mu+1)/n\sigma_n} t^{\nu} e^{-t^2/2}\,dt$$
$$= \sqrt{\frac{2}{\pi}}\sigma_n^{\nu}\int_{(c_n-1)/n\sigma_n}^{\infty} t^{\nu} e^{-t^2/2}\,dt$$

$$\leqslant \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\left(\frac{c_n-1}{n\sigma_n}\right)^2\right) \cdot \begin{cases} \dfrac{n\sigma_n}{c_n-1}, & \nu=0 \\ \sigma_n, & \nu=1 \\ \sigma_n^2\left(\dfrac{c_n-1}{n\sigma_n}+\dfrac{n\sigma_n}{c_n-1}\right), & \nu=2. \end{cases}$$

Now the assertions concerning $r_{\nu,n}^*$ for $\nu=0,1,2$ follow immediately by using Lemma 2; the estimate for $r_{4,n}^*$ follows analoguously.

(ii) With the notations above we have

$$\begin{aligned} &|F_n(f;x)-F_n^*(f;x)| \\ &\quad\leqslant \frac{M}{\sqrt{2\pi}\,n\sigma_n} \sum_{|j-nx|>c_n} \exp\left(K\left|\frac{j}{n}\right|-\frac{(j-nx)^2}{2n^2\sigma_n^2}\right) \\ &\quad\leqslant M\sqrt{\frac{2}{\pi}} \sum_{\xi_{jn}>a_n} \exp(K\sigma_n\xi_{j,n}+K|x|-\xi_{j,n}^2/2)\,(\xi_{j+1,n}-\xi_{j,n}) \\ &\quad= M\sqrt{\frac{2}{\pi}} \exp\left(K|x|+\frac{K^2\sigma_n^2}{2}\right) \\ &\qquad\times \sum_{\xi_{j,n}>a_n} \exp\left(-\frac{1}{2}(\xi_{j,n}-K\sigma_n)^2\right)(\xi_{j+1,n}-\xi_{j,n}) \\ &\quad\leqslant M\sqrt{\frac{2}{\pi}} \exp\left(K|x|+\frac{K^2\sigma_n^2}{2}\right) \\ &\qquad\times \int_{(c_n-1)/n\sigma_n}^{\infty} \exp\left(-\frac{1}{2}(t-K\sigma_n)^2\right)dt. \end{aligned}$$

Now a similar estimate as in the proof of part (i) completes the proof.

To have an idea of the concrete order of magnitudes in Lemma 3 again we exhibit the case (1.5) and choose

$$a_n=\bar{a}_n:=(\log n)^{1/2+\varepsilon}, \qquad \varepsilon>0, \quad n\geqslant 2, \tag{1.5$'$}$$

in (1.5). Using Lemma 2′ we obtain from Lemma 3

LEMMA 3′. (i) *For $x\in\mathbb{R}$ and $n\geqslant 8$ we have*

$$|r_{0,n}^*(x)| \leqslant \frac{3}{n^{2\pi^2}} + \sqrt{\frac{8}{\pi}}\frac{1}{\bar{a}_n}\exp(-\bar{a}_n^2/8) =: \alpha_n',$$

$$|r_{1,n}^*(x)| \leqslant \frac{13 \log n}{n^{2\pi^2+1}} + \sqrt{\frac{2 \log n}{\pi n^2}} \exp(-\bar{a}_n^2/8) =: \beta_n',$$

$$\left| r_{2,n}^*(x) - \frac{\log n}{n^2} \right|$$

$$\leqslant \frac{81 \log^2 n}{n^{2\pi^2+2}} + \sqrt{\frac{2}{\pi}} \frac{\log n}{n^2} \left(\bar{a}_n + \frac{2}{\bar{a}_n} \right) \exp(-\bar{a}_n^2/8) =: \gamma_n',$$

and

$$r_{4,n}^*(x) = O\left(\frac{\log^4 n}{n^8}\right) \qquad \textit{holds uniformly in } x \in \mathbb{R} \textit{ as } n \to \infty.$$

(ii) *If the function* $f: \mathbb{R} \to \mathbb{R}$ *satisfies the growth condition* (1.9), *then for all* $n \geqslant 8$ *with* $n/\sqrt{\log n} \geqslant 2K$ *and for all* $x \in \mathbb{R}$,

$$|F_n(f; x) - F_n^*(f; x)| \leqslant \sqrt{\frac{2}{\pi}} \frac{M e^{K|x| + 1/8}}{\bar{a}_n \left(1 - \frac{1}{\log n} - \frac{K}{n}\right)}$$

$$\times \exp\left(-\frac{\bar{a}_n^2}{2}\left(1 - \frac{1}{\log n} - \frac{K}{n}\right)^2\right).$$

2. Main Results

In this section we prove an explicit inequality for the approximation error and we determine the saturation class for F_n and F_n^*. For a compact interval I and a function $f \in C(I)$ we define the modulus of continuity by

$$\omega(f, \delta, I) := \sup_{\substack{x_1, x_2 \in I \\ |x_1 - x_2| < \delta}} |f(x_1) - f(x_2)|, \qquad \delta > 0. \tag{2.1}$$

THEOREM 1. *Suppose that the real function* f *is defined on* $I = [a, b]$, $a, b \in \mathbb{R}$, *that* $\alpha_n, \beta_n, \gamma_n$ *are defined as in Lemma* 3 *and the integer* $n_0(x)$ *is given by*

$$n_0(x) := \min\left\{ n \in \mathbb{N} \;\middle|\; \frac{c_n}{n} < \min(x - a, b - x) \right\} \qquad \textit{for } x \in (a, b).$$

(i) *If* $f \in C(I)$, $\sigma_n > 0$, $(c_n - 1)/n\sigma_n \geqslant \sqrt{2}$, *and* $n \geqslant n_0(x)$, *then for* $x \in (a, b)$

$$|F_n^*(f; x) - f(x)| \leqslant |f(x)|\, \alpha_n + 2(1 + \alpha_n)\, \omega(f, \sqrt{r_{2,n}^*(x)}, I), \tag{2.2}$$

where the asymptotic behaviour of $r_{2,n}^*(x)$ *is given in Lemma* 3(i).

(ii) *If* $f' \in C(I)$, $\sigma_n > 0$, $(c_n - 1)/n\sigma_n \geqslant \sqrt{2}$, *and* $n \geqslant n_0(x)$, *then for* $x \in (a, b)$

$$|F_n^*(f; x) - f(x)| \leqslant |f(x)|\, \alpha_n + |f'(x)|\, \beta_n + (2 + \alpha_n)\sqrt{r_{2,n}^*(x)}\, \omega(f', \sqrt{r_{2,n}^*(x)}, I). \qquad (2.3)$$

(iii) *If* $f \in C_2(I)$, $\sigma_n \to 0$, *and* $\alpha_n, \beta_n, \gamma_n = o(\sigma_n^2)$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2}(F_n^*(f; x) - f(x)) = \frac{f''(x)}{2} \qquad (0.6)^*$$

uniformly on any subset $[c, d] \subset I$ *with* $a < c < d < b$.

Proof. Parts (i) and (ii) are immediate consequences of Lemma 3(i) above and Theorem 2.3 in [8]. For part (iii) we write the Taylor series expansion of f at x

$$f(t) = f(x) + (t - x) f'(x) + \frac{(t - x)^2}{2} f''(x) + (t - x)^2 \eta(t - x),$$

where $|\eta(h)| < \varepsilon$ if $|h| < \delta$. Then we obtain

$$\begin{aligned} &F_n^*(f; x) - f(x) \\ &= \sum_{|j - nx| \leqslant c_n} \left(f\left(\frac{j}{n}\right) - f(x)\right) g_{jn}(x) + r_{0,n}^*(x) f(x) \\ &= r_{0,n}^*(x) f(x) + r_{1,n}^*(x) f'(x) + \frac{1}{2} r_{2,n}^*(x) f''(x) \\ &\quad + \sum_{|j - nx| \leqslant c_n} \left(\frac{j}{n} - x\right)^2 \eta\left(\frac{j}{n} - x\right) g_{jn}(x) \\ &= \frac{\sigma_n^2}{2} f''(x) + o(\sigma_n^2) \end{aligned}$$

by Lemma 3(i), the o-term being independent of $x \in [c, d]$.

Remarks. (i) It is obvious how to modify Theorem 1 in a local version. Furthermore Lemma 3(ii) implies that $(0.6)^*$ remains true with F_n in place of F_n^* in the presence of the growth condition (1.9) (note that $\alpha_n = o(\sigma_n^2)$).

(ii) Sufficient conditions for $\alpha_n, \beta_n, \gamma_n = o(\sigma_n^2)$ are

$$\sigma_n^2 \geqslant \frac{\log n}{\pi^2 n^2} \qquad \text{and} \qquad n\sigma_n \sqrt{\log \frac{1}{\sigma_n}} = o(c_n),$$

provided $\sigma_n \to 0$. This includes the classical Favard operators ($\sigma_n^2 = \lambda/2n$) and the optimal case.

The "Optimal" Case $\sigma_n^2 = \bar{\sigma}_n^2$ and $a_n = \bar{a}_n$

(iii) In this case we trivially have $\alpha_n', \beta_n', \gamma_n' = o(\log n/n^2)$ (see Lemma 3′(i)) and (0.6)* reads

$$\lim_{n\to\infty} \frac{n^2}{\log n} (F_n^*(f; x) - f(x)) = \frac{f''(x)}{2}. \tag{0.6$'$}$$

(iv) The "leading" terms in the explicit inequalities (2.2) and (2.3) are given by the right most terms; that is, we have from (2.2) and (2.3)

$$|F_n^*(f; x) - f(x)| \leqslant c_1 \omega \left(f; \frac{\sqrt{\log n}}{n}, I\right), \tag{2.2$'$}$$

and

$$|F_n^*(f; x) - f(x)| \leqslant c_2 \frac{\sqrt{\log n}}{n} \omega \left(f'; \frac{\sqrt{\log n}}{n}, I\right), \tag{2.3$'$}$$

if $f \in C(I)$ and $f' \in C(I)$, respectively, and $x \in (a, b)$, c_i are positive constants.

(v) It is well known (e.g., [13, p. 18]) for general operators of type (0.1) where $\{p_{jn}(x)\}_{-\infty}^{\infty}$ is the n-fold convolution of a probability distribution with expectation x such as Bernstein polynomials and similar operators that ($f \neq 0$ in a neighborhood of x)

$$\sum_j f\left(\frac{j}{n}\right) p_{jn}(x) = \sum_{|j-nx| \leqslant c_n} f\left(\frac{j}{n}\right) p_{jn}(x)(1 + o(1))$$

provided $c_n/\sqrt{n} \to \infty$ as $n \to \infty$. However, in our case $p_{jn} = g_{jn}$ with $\sigma_n^2 = \bar{\sigma}_n^2$ asymptotically "$(\log n)^{1+\varepsilon}$ terms" are sufficient. For practical purposes often it is enough to choose, for instance, $\varepsilon = 1/2$, 1, 2.

Next, we determine the saturation classes by the following saturation theorem. For notation see [8].

THEOREM 2. *Suppose that $I = [a, b]$, $a, b \in \mathbb{R}$, $f \in C(I)$ and α_n, β_n, γ_n (defined in Lemma 3) satisfy $\alpha_n, \beta_n, \gamma_n = o(\sigma_n^2)$ as $\sigma_n \to 0$. Then F_n^* is saturated on $I_1 = [c, d]$, $a < c < d < b$, with order σ_n^2, the trivial class*

$$T(F_n^*) := \{ g \in C(I_1) \mid g \text{ is linear on } I_1 \}$$

and saturation class

$$S(F_n^*) := \{ g \in C(I_1) \mid g' \in \mathrm{Lip}_1(I_1) \}.$$

Proof. The proof follows along standard lines using a general saturation theorem on linear positive operators [8, Theorem 5.3]. By Lemma 3(i) it follows that

$$\sup_{x \in \mathbb{R}} \frac{F_n^*(t^\nu; x) - x^\nu}{r_{2,n}^*(x)} = o(1), \qquad n \to \infty, \quad \nu = 0, 1,$$

and

$$\sup_{x \in \mathbb{R}} r_{4,n}^*(x) = O(\sigma_n^4), \qquad n \to \infty.$$

Now because of (0.6)* Theorem 2 follows from Theorem 5.3 in [8].

Remark. Since in Theorem 5.3, [6], only asymptotic conditions are involved, Theorem 2 carries over to F_n, if (1.9) holds.

References

1. V. A. Baskakov, An example of a sequence of linear positive operators in the space of continuous functions, *Dokl. Akad. Nauk. SSSR* **113** (1957), 249–251.
2. M. Becker, Inverse theorems for Favard operators in polynomial weight spaces, *Comment Math. Prace Mat.*, in press.
3. M. Becker, P. L. Butzer, and R. J. Nessel, Saturation for Favard operators in weighted function spaces, *Studia Math.* **59** (1976), 139–153.
4. S. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calul des probabilités, *Commun. Soc. Math. Kharkow 2* **13** (1912–1913), 1–2.
5. P. L. Butzer and R. J. Nessel, "Fourier Analysis and Approximation," Vol. I, Birkhäuser, Basel, 1971.
6. P. L. Butzer and M. Wehrens, Bernstein-type polynomials having $O(n^{-2})$ as saturation order, *Analysis*, in press.
7. R. Courant and D. Hilbert, "Methoden der mathematischen Physik I," Springer-Verlag, Berlin/Heidelberg/New York, 1968.
8. R. de Vore, "The Approximation of Continuous Functions by Positive Linear Operators," Lecture Notes in Mathematics No. 293, Springer-Verlag, Berlin/Heidelberg/New York, 1972.
9. J. Favard, Sur les multiplicateurs d'interpolation, *J. Math. Pures Appl.* **23** (1944), 219–247.
10. G. Felbecker, Linearkombinationen von iterierten Bernsteinoperatoren, *Manuscripta Math.* **29** (1979), 229–248.
11. W. Feller, "An Introduction to Probability Theory and Its Applications," Vol. II, Wiley, New York, 1971.
12. W. Gawronski, "Verallgemeinerte Bernsteinfunktionen und Schätzung einer Wahrscheinlichkeitsdichte," Habilitationsschrift, Ulm, 1980.

13. G. G. LORENTZ, "Bernstein polynomials," Univ. of Toronto Press, Toronto, 1953
14. C. MAY, Saturation and inverse theorems for combinations of a class of exponential type operators, *Canad. J. Math.* **28** (1978), 1224–1250.
15. M. W. MÜLLER, "Approximationstheorie," Akad. Verlagsgesellschaft, Wiesbaden, 1978.
16. J. J. SWETITS AND B. WOOD, A note on the degree of approximation with an optimal, discrete polynomial, *J. Approx. Theory* **28** (1980), 293–298.
17. O. SZÁSZ, Generalization of S. Bernstein's polynomials to the infinite interval, *J. Res. Nat. Bur. Standards* **45** (1950), 239–244.